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FINITE ELEMENT APPROXIMATION AND ITERATIVE METHOD SOLUTION OF ELLIPTIC CONTROL PROBLEM WITH CONSTRAINTS TO GRADIENT OF STATE[1)]

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КОНЕЧНО-ЭЛЕМЕНТНЫЕ АППРОКСИМАЦИИ И ИТЕРАЦИОННЫЙ МЕТОД РЕШЕНИЯ ЭЛЛИПТИЧЕСКИОЙ ЗАДАЧИ УПРАВЛЕНИЯ С ОГРАНИЧЕНИЯМИ НА ГРАДИЕНТ СОСТОЯНИЯ

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Summary

An optimal control problem with distributed control in the right-hand side of Poisson equation is considered. Pointwise constraints on the gradient of state and control are imposed in this problem. The convergence of finite element approximation for this problem is proved. Discrete saddle point problem is constructed and preconditioned Uzawa-type iterative algorithm for its solution is investigated.

Key words: optimal control, finite element method, iterative method, constrained saddle point problem

Аннотация

Изучается задача оптимального управления с распределенным управлением в правой части уравнения Пуассона. Накладываются поточечные ограничения как на управление, так и на градиент состояния. Доказана сходимость схемы метода конечных элементов. Дискретная задача формулируется в виде включения с седловым оператором, для которой исследуется сходимость итерационного метода типа Удзавы.

Ключевые слова: оптимальное управление, метод конечных элементов, итерационный метод, седловая задача с ограничениями

Introduction

An optimal control problem with distributed control in the right-hand side of Poisson equation is considered. Pointwise constraints on the gradient of state and control are imposed in this problem. The convergence of finite element approximation for this problem is proved. Discrete saddle point problem is constructed and preconditioned Uzawa-type iterative algorithm for its solution is investigated.

1. Optimal control problem and its approximation

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $\Omega_1 \subseteq \Omega$ be its polygonal subdomain. Define arbitrary functions $y_d, u_d \in L_2(\Omega)$, and

$$\text{functions } y^*, u_1^*, u_2^* \text{ from } C(\overline{\Omega}),\ \text{such that } y^*(x) > 0,\ u_1^*(x) < 0 < u_2^*(x) \text{ at } x \in \overline{\Omega}. \tag{1}$$

Let state problem is the Dirichlet problem for the Poisson equation:

$$y \in H_0^1(\Omega) : \int_\Omega \nabla y \cdot \nabla z \, dx = \int_\Omega uz dx \ \ \forall\, z \in H_0^1(\Omega), \tag{2}$$

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where $u(x)$ is the control function and solution $y(x)$ of equation (2) is state of the system. Define the convex and closed sets of the constraints for control and state functions:

$$U_{ad} = \{u \in L_2(\Omega) : u_1^*(x) \leqslant u(x) \leqslant u_2^*(x) \text{ a.e. in } \Omega\},$$
$$Y_{ad} = \{y \in H_0^1(\Omega) : |\nabla y(x)| \leqslant y^*(x) \text{ a.e. in } \Omega\}.$$

Consider the following optimal control problem:

$$\min_{(y,u)\in K} \Big\{ J(y,u) = \frac{1}{2}\int_{\Omega_1} (y-y_d)^2\,dx + \frac{\alpha}{2}\int_{\Omega} (u-u_d)^2\,dx \Big\},\ \alpha = const > 0,$$
$$K = \{(y,u) \in Y_{ad}\times U_{ad} \text{ and } y \text{ satisfy equation (2)}\}. \tag{3}$$

Lemma 1. *Problem* (3) *has a unique solution.*

Below we use the notation $\|\cdot\|_{0,p}$ for norms of Lebesgue spaces $L_p(\Omega)$ and $\|\cdot\|_{l,p}$ for norms of Sobolev spaces $W_p^l(\Omega)$ for $1 \leqslant p \leqslant \infty$ and integers $l > 0$.

Let $\mathcal{T}_h = \bigcup e_i$ be a conforming and regular triangulation of the domain Ω, h be the maximum diameter of elements $e \in \mathcal{T}_h$. We assume that the triangulation is compatible with Ω_1 in the sense that $\overline{\Omega}_1$ consists of a number of triangles $e \in \mathcal{T}_{1h} \subseteq \mathcal{T}_h$. We define the finite element spaces $H_h = \{y_h \in H_0^1(\Omega) : y_h(x) \in P_1 \text{ on } e \in \mathcal{T}_h\}$, $U_h = \{u_h \in L_2(\Omega) : u_h(x) \in P_0 \text{ on } e \in \mathcal{T}_h\}$, where P_k is the set of polynomials of degree at most k in all variables. We denote by π_h the operator of integral averaging of functions from $L_1(\Omega)$, with values in U_h:

$$\pi_h u(x) = |e_i|^{-1}\int_{e_i} u(t)dt \ \text{ for } \ x \in e_i, \quad |e_i| = \text{meas}\, e_i.$$

Let $y_{dh} = \pi_h y_d$, $u_{dh} = \pi_h u_d$, $y_h^* = \pi_h y^*$, $u_{1h}^* = \pi_h u_1^*$, $u_{2h}^* = \pi_h u_2^*$. Then $y_h^*(x) > 0$ and $u_{1h}^*(x) < 0 < u_{2h}^*(x)$. We define a convex and closed sets of the constraints on the mesh control and state functions: $Y_{ad}^h = \{y_h \in H_h : |\nabla y_h| \leqslant y_h^* \text{ on } \Omega\}$, $U_{ad}^h = \{u_h \in U_h : u_{1h}^* \leqslant u_h \leqslant u_{2h}^* \text{ on } \Omega\}$. Discrete state problem is the approximation by the finite element method of the boundary value problem (2):

$$y_h \in H_h : \int_{\Omega} \nabla y_h \cdot \nabla z_h\,dx = \int_{\Omega} u_h z_h\,dx \ \ \forall\, z_h \in H_h, \quad u_h \in U_h. \tag{4}$$

Objective function $J_h : H_h \times U_h \to \mathbb{R}$ is defined by the equality

$$J_h(y_h,u_h) = \frac{1}{2}\int_{\Omega_1} (y_h - y_{dh})^2\,dx + \frac{\alpha}{2}\int_{\Omega} (u_h - u_{dh})^2\,dx.$$

It is easy to verify that the discrete optimal control problem

$$\min_{(y_h,u_h)\in K_h} J_h(y_h,u_h),$$
$$K_h = \{(y_h,u_h) \in Y_{ad}^h \times U_{ad}^h \text{ and } y_h \text{ is a solution of (4)}\} \tag{5}$$

has a unique solution (y_h,u_h). The reasoning is the same as that for problem (3), namely, set K_h is a nonempty convex compact, and the function J_h is continuous and strictly convex on K_h.

By using the traditional approach to the study of the convergence of discrete approximations for variational inequalities and minimization problems we prove

Theorem 1. *Solutions* $\{(y_h,u_h)\}$ *of the problem* (5) *strongly converge to the solution* (y,u) *of* (3) *in* $H_0^1(\Omega)\times L_2(\Omega)$ *when* $h \to 0$.

2. Discrete saddle point problem

Introduce an auxiliary function $\bar{p}_h = \nabla y_h \in U_h \times U_h$ and define a set of constraints $P_{ad}^h = \{\bar{p}_h \in U_h \times U_h : |\bar{p}_h(x)| \leqslant y_h^*(x)$ a.e. in $\Omega\}$. Then problem (5) can be rewritten as follows:

$$\min_{(y_h, u_h, \bar{p}_h) \in W_h} \Big\{ J_h(y_h, u_h) = \frac{1}{2} \int\limits_{\Omega_1} (y_h - y_{dh})^2 \, dx + \frac{\alpha}{2} \int\limits_{\Omega} (u_h - u_{dh})^2 \, dx \Big\}, \tag{6}$$

$$W_h = \{(y_h, u_h, \bar{p}_h) \in H_h \times P_{ad}^h \times U_{ad}^h,\ \bar{p}_h = \nabla y_h,\ y_h \text{ is a solution of } (4)\}.$$

Let Lagrangian function be defined by the equality

$$\mathcal{L}_h(y_h, u_h, \bar{p}_h, \lambda_h, \bar{\mu}_h) = J_h(y_h, u_h) + \int\limits_{\Omega} \nabla y_h \cdot \nabla \lambda_h \, dx -$$

$$- \int\limits_{\Omega} u_h \lambda_h \, dx + \int\limits_{\Omega} \bar{\mu}_h (\nabla y_h - \bar{p}_h) \, dx, \tag{7}$$

where the Lagrange multipliers $\lambda_h \in H_h,\ \bar{\mu}_h \in U_h \times U_h$, and the saddle point are looking under constraints on direct variables $\bar{p}_h \in P_{ad}^h, u_h \in U_{ad}^h$.

For further formulation the saddle point problem in algebraic form we assign to the functions of the finite element spaces H_h and U_h the vectors of their nodal parameters. Let $\omega_h = \{t_i\}_{i=1}^m$ be the set of vertices of triangles $e \in T_h$, lying in Ω, $m = \operatorname{card} \omega_h$, $\xi_h = \{t_i\}_{i=1}^s$ be the set of barycenters of the triangles $e \in T_h$. Put in correspondence function $y_h \in H_h$ and vector $y \in \mathbb{R}^m$ with coordinates $y_i = y_h(t_i),\ t_i \in \omega_h$ (with any node numbering t_i), and the functions $u_h \in U_h$ – vector $u \in \mathbb{R}^s$ with coordinates $u_i = u_h(t_i),\ t_i \in \xi_h$. We will use the notation $y \Leftrightarrow y_h$, $u \Leftrightarrow u_h$.

Define the matrices $L \in \mathbb{R}^{m \times m}$, $M_u \in \mathbb{R}^{s \times s}$, $M_y \in \mathbb{R}^{m \times m}$, $S \in \mathbb{R}^{s \times m}$, $R_i \in \mathbb{R}^{m \times s}$ $(i = 1, 2)$ by the equalities:

$$(Ly, z) = \int\limits_{\Omega} \nabla y_h \cdot \nabla z_h \, dx, \quad (M_u u, v) = \int\limits_{\Omega} u_h(x) v_h(x) \, dx, \quad (M_y y, z) = \int\limits_{\Omega_1} y_h z_h \, dx,$$

$$(R_i y, v) = \int\limits_{\Omega} \frac{\partial y_h}{\partial x_i}(x) v_h(x) \, dx, \quad (Su, y) = \int\limits_{\Omega} u_h(x) y_h(x) \, dx, \quad (S_1 u, y) = \int\limits_{\Omega_1} u_h(x) y_h(x) \, dx.$$

These equalities must be satisfied for all $y, z \in \mathbb{R}^m$ and $u, v \in \mathbb{R}^s$. By construction, M_u is a diagonal positive definite matrix.

Lagrange function (7) and a sets of constraints in terms of vectors of nodal parameters of mesh functions take the form:

$$\mathcal{L}(y, u, \bar{p}, \lambda, \bar{\mu}) = \frac{1}{2} (M_y y, y) + (S_1 y_d, y) + \frac{\alpha}{2} (M_u(u - u_d), u - u_d) +$$

$$+ (Ly - Su, \lambda) + (Ry - \overline{M}_u \bar{p}, \bar{\mu}),$$

$$P_{ad} = \{\bar{p} \in \mathbb{R}^s \times \mathbb{R}^s : p_{1j}^2 + p_{2j}^2 \leqslant y_j^{*2} \text{ for all } j = 1, 2, \ldots, s\},$$

$$U_{ad} = \{u \in \mathbb{R}^s : u_i \in [u_{1i}^*, u_{2i}^*] \text{ for all } i = 1, 2, \ldots, m\}.$$

Let $\varphi_p(\bar{p})$ and $\varphi_u(u)$ be the indicator functions of the sets P_{ad} and U_{ad}. Then the corresponding saddle point problem lead to the system

$$\begin{pmatrix} M_y + rL & 0 & -rS & L & R^T \\ -rR & r\overline{M}_u & 0 & 0 & -\overline{M}_u \\ 0 & 0 & \alpha M_u & -S^T & 0 \\ L & 0 & -S & 0 & 0 \\ R & -\overline{M}_u & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ \bar{p} \\ u \\ \lambda \\ \bar{\mu} \end{pmatrix} + \begin{pmatrix} -S_1 y_d \\ \partial \varphi_p(\bar{p}) \\ \partial \varphi_u(u) - M_u u_d \\ 0 \\ 0 \end{pmatrix} \ni 0. \tag{8}$$

With the notations

$$A = \begin{pmatrix} M_y + rL & 0 & -rS \\ -rR & r\overline{M}_u & 0 \\ 0 & 0 & \alpha M_u \end{pmatrix}, \quad B = \begin{pmatrix} L & 0 & -S \\ R & -\overline{M}_u & 0 \end{pmatrix}, \quad D = \begin{pmatrix} L & 0 \\ 0 & \overline{M}_u \end{pmatrix},$$

$$x = (y, \bar{p}, u)^T, \quad \eta = (\lambda, \quad \bar{\mu})^T, \quad f = (M_y y_d, 0, M_u u_d)^T, \quad \varphi(x) = \varphi_u(u) + \varphi_p(\bar{p})$$

problem (8) can be written as

$$\begin{pmatrix} A & -B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} \partial\varphi(x) \\ 0 \end{pmatrix} \ni \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{9}$$

We assume that the parameter r is chosen so that $0 < r < 3\alpha/c_f^2$, where c_f − constant in the Friedrichs inequality. Then the matrix A is positive definite. In its turn, the matrix B has full column rank, since its block $\begin{pmatrix} L & 0 \\ R & -\overline{M}_u \end{pmatrix}$ is a nonsingular matrix. Vector with coordinates $u = 0, \bar{p} = 0, y = 0$ belongs to interior of the constraint sets, as well as to the kernel of matrix B. This implies the existence of a solution $(y, \bar{p}, u, \lambda, \bar{\mu})$ to problem (8) with unique $(y, \bar{p}, u)$ (the components $\eta = (\lambda, \mu)$ of the solution are not uniquely defined). Corresponding to the vector $(y, \bar{p}, u)$ mesh function $(y_h, \bar{p}_h, u_h)$ coincides with the solution of the discrete optimal control problem (6).

3. Preconditioned Uzawa-type iterative method.

From the system (9) we obtain the equation $B(A + \partial\varphi)^{-1}(B^T\eta + f) = 0$ for $\eta = (\lambda, \bar{\mu})^T$. To solve it we apply one-step iterative method

$$\frac{1}{\tau} D(\eta^{k+1} - \eta^k) + B(A + \partial\varphi)^{-1}(B^T\eta^k + f) = 0. \tag{10}$$

Let $m(r) > 0$ is the minimum eigenvalue of K_r,

$$K_r = \begin{pmatrix} r & -0.5r & -0.5rc_f \\ -0.5r & r & 0 \\ -0.5rc_f & 0 & \alpha \end{pmatrix}.$$

Theorem 2. *Let* $0 < r < 3\alpha/c_f^2$. *Then Uzawa method* (10) *for problem* (8) *converges if*

$$0 < \tau < \frac{2m(r)}{\max\{2 + c_f^2, 3\}}.$$

In proving the theorem 2 we use the results of [2] and [3].

4. Implementation of the Preconditioned Uzawa method.

It is easy to see that one iteration of method (10) reduces to implementation of the following calculations for the known λ^k и $\bar{\mu}^k$:

1. $u^{k+1} = (\alpha M_u + \partial\varphi_u)^{-1}(S^T\lambda^k + M_u u_d) = Pr_{U_{ad}}(\alpha^{-1}M_u^{-1}(S^T\lambda^k + M_u u_d)$;
2. $y^{k+1} = (M_y + rL)^{-1}(S_1 y_d + rSu^{k+1} - L\lambda^k - R^T\bar{\mu}^k)$;
3. $\bar{p}^{k+1} = (r\overline{M}_u + \partial\varphi_p)^{-1}(\overline{M}_u\bar{\mu}^k + rRy^{k+1}) = Pr_{P_{ad}}(r^{-1}\bar{\mu}^k + \overline{M}_u^{-1}Ry^{k+1})$;
4. $\lambda^{k+1} = \lambda^k + \tau(y^{k+1} - L^{-1}Su^{k+1})$;
5. $\bar{\mu}^{k+1} = \bar{\mu}^k + \tau(\overline{M}_u^{-1}Ry^{k+1} - \bar{p}^{k+1})$.

By virtue of diagonality of the matrices M_u and $\overline{M}_u = \mathrm{diag}\,(M_u, M_u)$ and pointwise constraints for $u \in U_{ad}$ and $\bar{p} \in P_{ad}$ the determination of u^{k+1} and $\bar{p}^{k+1}$ reduces to the pointwise projections of known vectors to the corresponding sets of constraints. More precisely, for a fixed i: $u_i^{k+1} = Pr_{[-u_{1i}^*, u_{2i}^*]}\Big((\alpha m_{ii})^{-1}\,(S^T\lambda^k + M_u u_d)_i\Big)$, where m_{ii} is a diagonal element M_u, and

$$|\bar{p}_i^{k+1}| = Pr_{[0,y_i^*]}|\bar{F}|, \quad p_{i1}^{k+1} = |\bar{p}_i^{k+1}|^{-1}F_1, \quad p_{i2}^{k+1} = |\bar{p}_i^{k+1}|^{-1}F_2,$$

where $\bar{F} = (F_1, F_2) = (r^{-1}\bar{\mu}^k + \overline{M}_u^{-1}Ry^{k+1})_i$.

5. Control of accuracy and stopping criterion.

When the saddle point problem (9) is solved by any iterative method, we find not only an approximation of (x^k, η^k) to the exact solution (x, η), but also the vector $\gamma^k \in \partial\varphi(x^k)$ — the unique selection from the set $\partial\varphi(x^k)$. We define the components of the residual vector by the equalities $r_x^k = f - Ax^k - \gamma^k + B^T\eta^k$, $r_\eta^k = -Bx^k$. Then the error vector $(x - x^k, \eta - \eta^k)^T$ satisfies the system

$$\begin{pmatrix} A & -B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} x - x^k \\ \eta - \eta^k \end{pmatrix} + \begin{pmatrix} \partial\varphi(x) - \gamma^k \\ 0 \end{pmatrix} \ni \begin{pmatrix} r_x^k \\ r_\eta^k \end{pmatrix}.$$

Multiplying this system scalarly by the vector $(x - x^k, \eta - \eta^k)^T$ and applying the inequality $(\partial\varphi(x) - \partial\varphi(x^k), x - x^k) \geqslant 0$, we get $(A(x - x^k), x - x^k) \leqslant (r_x^k, x - x^k) + (r_\eta^k, \eta - \eta^k)$. Hence

$$\|x - x^k\|_{A_s}^2 \leqslant \|r_x^k\|_{A_s^{-1}}\|x - x^k\|_{A_s} + |(r_\eta^k, \eta - \eta^k)|. \tag{11}$$

Since the inclusion $Ax - B^T\eta + \partial\varphi(x) \ni f$ is solved exactly at each iteration of Uzawa method (10), therefore $r_x^k = 0$ and estimate (11) takes the form

$$\|x - x^k\|_{A_s} \leqslant |(r_\eta^k, \eta - \eta^k)| \leqslant \|\eta - \eta^k\|_D^{1/2}\|r_\eta^k\|_{D^{-1}}^{1/2} \tag{12}$$

where D is the preconditioner of this method. Since $\|\eta - \eta^k\|_D \to 0$ for $k \to \infty$, inequality (12) gives the information about error $\|x - x^k\|_{A_s}$ through the estimate of the norm of the residual component $\|r_\eta^k\|_{D^{-1}}$, namely, $\|x - x^k\|_{A_s} = o(\|r_\eta^k\|_{D^{-1}}^{1/2})$ when $k \to \infty$. In the problem (8) vector $r_\eta^k = (Ly^k - Su^k, Ry^k - \overline{M}_u\bar{p}^k)$, so the upper bound for number of iterations is the value

$$\delta^k = \|r_\eta^k\|_{D^{-1}}^{1/2} = \big((Ly^k - Su^k, y^k - L^{-1}Su^k) + (Ry^k - \overline{M}_u\bar{p}^k, \overline{M}_u^{-1}Ry^k - \bar{p}^k)\big)^{1/2}.$$

Note that the vectors

$$Ly^k - Su^k, \quad y^k - L^{-1}Su^k = (\lambda^k - \lambda^{k-1})/\tau, \quad Ry^k - \overline{M}_u\bar{p}^k, \quad M_u^{-1}Ry^k - \bar{p}^k = (\bar{\mu}^{k+1} - \bar{\mu}^k)/\tau,$$

are computed when implementing the algorithm, thus, control of the value δ^k does not lead to additional computational cost.

R E F E R E N C E S

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